CITED REFERENCES AND FURTHER READING:

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapters 6, 7, and 26.

Pearson, K. (ed.) 1951, *Tables of the Incomplete Gamma Function* (Cambridge: Cambridge University Press).

6.3 Exponential Integrals

The standard definition of the exponential integral is

$$E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n} dt, \qquad x > 0, \quad n = 0, 1, \dots \tag{6.3.1}$$

The function defined by the principal value of the integral

$$\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{e^{-t}}{t}\, dt = \int_{-\infty}^{x} \frac{e^t}{t}\, dt, \qquad x > 0 \tag{6.3.2}$$

is also called an exponential integral. Note that $\mathrm{Ei}(-x)$ is related to $-E_1(x)$ by analytic continuation.

The function $E_n(x)$ is a special case of the incomplete gamma function

$$E_n(x) = x^{n-1}\Gamma(1-n, x) \tag{6.3.3}$$

We can therefore use a similar strategy for evaluating it. The continued fraction — just equation (6.2.6) rewritten — converges for all $x > 0$:

$$E_n(x) = e^{-x}\left(\frac{1}{x+}\,\frac{n}{1+}\,\frac{1}{x+}\,\frac{n+1}{1+}\,\frac{2}{x+}\cdots\right) \tag{6.3.4}$$

We use it in its more rapidly converging even form,

$$E_n(x) = e^{-x}\left(\frac{1}{x+n-}\,\frac{1\cdot n}{x+n+2-}\,\frac{2(n+1)}{x+n+4-}\cdots\right) \tag{6.3.5}$$

The continued fraction only really converges fast enough to be useful for $x \gtrsim 1$. For $0 < x \lesssim 1$, we can use the series representation

$$E_n(x) = \frac{(-x)^{n-1}}{(n-1)!}[-\ln x + \psi(n)] - \sum_{\substack{m=0 \\ m\neq n-1}}^{\infty} \frac{(-x)^m}{(m-n+1)m!} \tag{6.3.6}$$

The quantity $\psi(n)$ here is the digamma function, given for integer arguments by

$$\psi(1) = -\gamma, \qquad \psi(n) = -\gamma + \sum_{m=1}^{n-1} \frac{1}{m} \tag{6.3.7}$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant. We evaluate the expression (6.3.6) in order of ascending powers of x:

$$E_n(x) = -\left[\frac{1}{(1-n)} - \frac{x}{(2-n)\cdot 1} + \frac{x^2}{(3-n)(1\cdot 2)} - \cdots + \frac{(-x)^{n-2}}{(-1)(n-2)!}\right] + \frac{(-x)^{n-1}}{(n-1)!}[-\ln x + \psi(n)] - \left[\frac{(-x)^n}{1\cdot n!} + \frac{(-x)^{n+1}}{2\cdot(n+1)!} + \cdots\right] \tag{6.3.8}$$

The first square bracket is omitted when $n = 1$. This method of evaluation has the advantage that for large n the series converges before reaching the term containing $\psi(n)$. Accordingly, one needs an algorithm for evaluating $\psi(n)$ only for small n, $n \lesssim 20-40$. We use equation (6.3.7), although a table look-up would improve efficiency slightly.

Amos [1] presents a careful discussion of the truncation error in evaluating equation (6.3.8), and gives a fairly elaborate termination criterion. We have found that simply stopping when the last term added is smaller than the required tolerance works about as well.

Two special cases have to be handled separately:

$$\begin{aligned} E_0(x) &= \frac{e^{-x}}{x} \\ E_n(0) &= \frac{1}{n-1}, \qquad n > 1 \end{aligned} \tag{6.3.9}$$

The routine `expint` allows fast evaluation of $E_n(x)$ to any accuracy `EPS` within the reach of your machine's word length for floating-point numbers. The only modification required for increased accuracy is to supply Euler's constant with enough significant digits. Wrench [2] can provide you with the first 328 digits if necessary!

```
#include <math.h>
#define MAXIT 100                Maximum allowed number of iterations.
#define EULER 0.5772156649       Euler's constant γ.
#define FPMIN 1.0e-30            Close to smallest representable floating-point number.
#define EPS 1.0e-7               Desired relative error, not smaller than the machine pre-
                                    cision.
float expint(int n, float x)
Evaluates the exponential integral En(x).
{
    void nrerror(char error_text[]);
    int i,ii,nm1;
    float a,b,c,d,del,fact,h,psi,ans;

    nm1=n-1;
    if (n < 0 || x < 0.0 || (x==0.0 && (n==0 || n==1)))
    nrerror("bad arguments in expint");
    else {
        if (n == 0) ans=exp(-x)/x;                Special case.
        else {
            if (x == 0.0) ans=1.0/nm1;            Another special case.

            else {
```

```
                if (x > 1.0) {                          Lentz's algorithm (§5.2).
                    b=x+n;
                    c=1.0/FPMIN;
                    d=1.0/b;
                    h=d;
                    for (i=1;i<=MAXIT;i++) {
                        a = -i*(nm1+i);
                        b += 2.0;
                        d=1.0/(a*d+b);                  Denominators cannot be zero.
                        c=b+a/c;
                        del=c*d;
                        h *= del;
                        if (fabs(del-1.0) < EPS) {
                            ans=h*exp(-x);
                            return ans;
                        }
                    }
                    nrerror("continued fraction failed in expint");
                } else {                                Evaluate series.
                    ans = (nm1!=0 ? 1.0/nm1 : -log(x)-EULER);       Set first term.
                    fact=1.0;
                    for (i=1;i<=MAXIT;i++) {
                        fact *= -x/i;
                        if (i != nm1) del = -fact/(i-nm1);
                        else {
                            psi = -EULER;               Compute ψ(n).
                            for (ii=1;ii<=nm1;ii++) psi += 1.0/ii;
                            del=fact*(-log(x)+psi);
                        }
                        ans += del;
                        if (fabs(del) < fabs(ans)*EPS) return ans;
                    }
                    nrerror("series failed in expint");
                }
            }
        }
    }
    return ans;
}
```

A good algorithm for evaluating Ei is to use the power series for small x and the asymptotic series for large x. The power series is

$$\mathrm{Ei}(x) = \gamma + \ln x + \frac{x}{1 \cdot 1!} + \frac{x^2}{2 \cdot 2!} + \cdots \tag{6.3.10}$$

where γ is Euler's constant. The asymptotic expansion is

$$\mathrm{Ei}(x) \sim \frac{e^x}{x}\left(1 + \frac{1!}{x} + \frac{2!}{x^2} + \cdots\right) \tag{6.3.11}$$

The lower limit for the use of the asymptotic expansion is approximately $|\ln \mathtt{EPS}|$, where EPS is the required relative error.

```
#include <math.h>
#define EULER 0.57721566          Euler's constant γ.
#define MAXIT 100                 Maximum number of iterations allowed.
#define FPMIN 1.0e-30             Close to smallest representable floating-point number.
#define EPS 6.0e-8                Relative error, or absolute error near the zero of Ei at
                                      x = 0.3725.

float ei(float x)
Computes the exponential integral Ei(x) for x > 0.
{
    void nrerror(char error_text[]);
    int k;
    float fact,prev,sum,term;

    if (x <= 0.0) nrerror("Bad argument in ei");
    if (x < FPMIN) return log(x)+EULER;       Special case: avoid failure of convergence
    if (x <= -log(EPS)) {                         test because of underflow.
        sum=0.0;                              Use power series.
        fact=1.0;
        for (k=1;k<=MAXIT;k++) {
            fact *= x/k;
            term=fact/k;
            sum += term;
            if (term < EPS*sum) break;
        }
        if (k > MAXIT) nrerror("Series failed in ei");
        return sum+log(x)+EULER;
    } else {                                  Use asymptotic series.
        sum=0.0;                              Start with second term.
        term=1.0;
        for (k=1;k<=MAXIT;k++) {
            prev=term;
            term *= k/x;
            if (term < EPS) break;
            Since final sum is greater than one, term itself approximates the relative error.
            if (term < prev) sum += term;     Still converging: add new term.
            else {
                sum -= prev;                  Diverging: subtract previous term and
                break;                            exit.
            }
        }
        return exp(x)*(1.0+sum)/x;
    }
}
```

CITED REFERENCES AND FURTHER READING:

Stegun, I.A., and Zucker, R. 1974, *Journal of Research of the National Bureau of Standards*, vol. 78B, pp. 199–216; 1976, *op. cit.*, vol. 80B, pp. 291–311.

Amos D.E. 1980, *ACM Transactions on Mathematical Software*, vol. 6, pp. 365–377 [1]; also vol. 6, pp. 420–428.

Abramowitz, M., and Stegun, I.A. 1964, *Handbook of Mathematical Functions*, Applied Mathematics Series, Volume 55 (Washington: National Bureau of Standards; reprinted 1968 by Dover Publications, New York), Chapter 5.

Wrench J.W. 1952, *Mathematical Tables and Other Aids to Computation*, vol. 6, p. 255. [2]

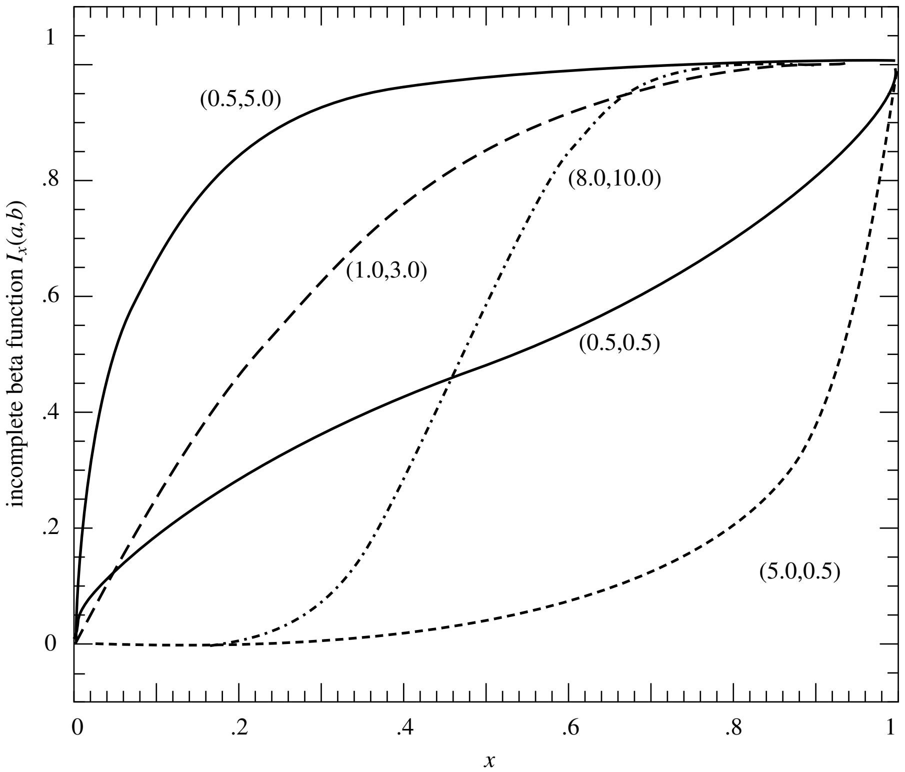

Figure 6.4.1. The incomplete beta function $I_x(a,b)$ for five different pairs of (a,b). Notice that the pairs $(0.5, 5.0)$ and $(5.0, 0.5)$ are related by reflection symmetry around the diagonal (cf. equation 6.4.3).

6.4 Incomplete Beta Function, Student's Distribution, F-Distribution, Cumulative Binomial Distribution

The incomplete beta function is defined by

$$I_x(a,b) \equiv \frac{B_x(a,b)}{B(a,b)} \equiv \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}dt \qquad (a,b>0) \tag{6.4.1}$$

It has the limiting values

$$I_0(a,b) = 0 \qquad I_1(a,b) = 1 \tag{6.4.2}$$

and the symmetry relation

$$I_x(a,b) = 1 - I_{1-x}(b,a) \tag{6.4.3}$$

If a and b are both rather greater than one, then $I_x(a,b)$ rises from "near-zero" to "near-unity" quite sharply at about $x = a/(a+b)$. Figure 6.4.1 plots the function for several pairs (a,b).