`void pcshft(float a, float b, float d[], int n)`
Polynomial coefficient shift. Given a coefficient array `d[0..n-1]`, this routine generates a coefficient array g`[0..n-1]` such that $\sum_{k=0}^{n-1} d_k y^k = \sum_{k=0}^{n-1} g_k x^k$, where x and y are related by (5.8.10), i.e., the interval $-1 < y < 1$ is mapped to the interval $a < x < b$. The array g is returned in d.

```
{
    int k,j;
    float fac,cnst;

    cnst=2.0/(b-a);
    fac=cnst;
    for (j=1;j<n;j++) {             First we rescale by the factor const...
        d[j] *= fac;
        fac *= cnst;
    }
    cnst=0.5*(a+b);                 ...which is then redefined as the desired shift.
    for (j=0;j<=n-2;j++)            We accomplish the shift by synthetic division. Synthetic
        for (k=n-2;k>=j;k--)           division is a miracle of high-school algebra. If you
            d[k] -= cnst*d[k+1];       never learned it, go do so. You won't be sorry.
}
```

CITED REFERENCES AND FURTHER READING:

Acton, F.S. 1970, *Numerical Methods That Work*; 1990, corrected edition (Washington: Mathematical Association of America), pp. 59, 182–183 [synthetic division].

5.11 Economization of Power Series

One particular application of Chebyshev methods, the *economization of power series*, is an occasionally useful technique, with a flavor of getting something for nothing.

Suppose that you are already computing a function by the use of a convergent power series, for example

$$f(x) \equiv 1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \cdots \tag{5.11.1}$$

(This function is actually $\sin(\sqrt{x})/\sqrt{x}$, but pretend you don't know that.) You might be doing a problem that requires evaluating the series many times in some particular interval, say $[0, (2\pi)^2]$. Everything is fine, except that the series requires a large number of terms before its error (approximated by the first neglected term, say) is tolerable. In our example, with $x = (2\pi)^2$, the first term smaller than 10^{-7} is $x^{13}/(27!)$. This then approximates the error of the finite series whose last term is $x^{12}/(25!)$.

Notice that because of the large exponent in x^{13}, the error is *much smaller* than 10^{-7} everywhere in the interval except at the very largest values of x. This is the feature that allows "economization": if we are willing to let the error elsewhere in the interval rise to about the same value that the first neglected term has at the extreme end of the interval, then we can replace the 13-term series by one that is significantly shorter.

Here are the steps for doing so:

1. Change variables from x to y, as in equation (5.8.10), to map the x interval into $-1 \le y \le 1$.
2. Find the coefficients of the Chebyshev sum (like equation 5.8.8) that exactly equals your truncated power series (the one with enough terms for accuracy).
3. Truncate this Chebyshev series to a smaller number of terms, using the coefficient of the first neglected Chebyshev polynomial as an estimate of the error.

4. Convert back to a polynomial in y.
5. Change variables back to x.

All of these steps can be done numerically, given the coefficients of the original power series expansion. The first step is exactly the inverse of the routine pcshft (§5.10), which mapped a polynomial from y (in the interval $[-1,1]$) to x (in the interval $[a,b]$). But since equation (5.8.10) is a linear relation between x and y, one can also use pcshft for the inverse. The inverse of

$$\texttt{pcshft}(a,b,\texttt{d},\texttt{n})$$

turns out to be (you can check this)

$$\texttt{pcshft}\left(\frac{-2-b-a}{b-a},\frac{2-b-a}{b-a},\texttt{d},\texttt{n}\right)$$

The second step requires the inverse operation to that done by the routine chebpc (which took Chebyshev coefficients into polynomial coefficients). The following routine, pccheb, accomplishes this, using the formula [1]

$$x^k = \frac{1}{2^{k-1}}\left[T_k(x) + \binom{k}{1}T_{k-2}(x) + \binom{k}{2}T_{k-4}(x) + \cdots\right] \tag{5.11.2}$$

where the last term depends on whether k is even or odd,

$$\cdots + \binom{k}{(k-1)/2}T_1(x) \quad (k \text{ odd}), \qquad \cdots + \frac{1}{2}\binom{k}{k/2}T_0(x) \quad (k \text{ even}). \tag{5.11.3}$$

```
void pccheb(float d[], float c[], int n)
Inverse of routine chebpc: given an array of polynomial coefficients d[0..n-1], returns an
equivalent array of Chebyshev coefficients c[0..n-1].
{
    int j,jm,jp,k;
    float fac,pow;

    pow=1.0;                         Will be powers of 2.
    c[0]=2.0*d[0];
    for (k=1;k<n;k++) {              Loop over orders of x in the polynomial.
        c[k]=0.0;                    Zero corresponding order of Chebyshev.
        fac=d[k]/pow;
        jm=k;
        jp=1;
        for (j=k;j>=0;j-=2,jm--,jp++) {
        Increment this and lower orders of Chebyshev with the combinatorial coefficent times
        d[k]; see text for formula.
            c[j] += fac;
            fac *= ((float)jm)/((float)jp);
        }
        pow += pow;
    }
}
```

The fourth and fifth steps are accomplished by the routines chebpc and pcshft, respectively. Here is how the procedure looks all together:

```
#define NFEW ..
#define NMANY ..

    float *c,*d,*e,a,b;
    Economize NMANY power series coefficients e[0..NMANY-1] in the range (a,b) into NFEW
    coefficients d[0..NFEW-1].

    c=vector(0,NMANY-1);
    d=vector(0,NFEW-1);
    e=vector(0,NMANY-1);
    pcshft((-2.0-b-a)/(b-a),(2.0-b-a)/(b-a),e,NMANY);
    pccheb(e,c,NMANY);
    ...
    Here one would normally examine the Chebyshev coefficients c[0..NMANY-1] to decide
    how small NFEW can be.
    chebpc(c,d,NFEW);
    pcshft(a,b,d,NFEW);
```

In our example, by the way, the 8th through 10th Chebyshev coefficients turn out to be on the order of -7×10^{-6}, 3×10^{-7}, and -9×10^{-9}, so reasonable truncations (for single precision calculations) are somewhere in this range, yielding a polynomial with 8 – 10 terms instead of the original 13.

Replacing a 13-term polynomial with a (say) 10-term polynomial without any loss of accuracy — that does seem to be getting something for nothing. Is there some magic in this technique? Not really. The 13-term polynomial defined a function $f(x)$. Equivalent to economizing the series, we could instead have evaluated $f(x)$ at enough points to construct its Chebyshev approximation in the interval of interest, by the methods of §5.8. We would have obtained just the same lower-order polynomial. The principal lesson is that the rate of convergence of Chebyshev coefficients has nothing to do with the rate of convergence of power series coefficients; and it is the *former* that dictates the number of terms needed in a polynomial approximation. A function might have a *divergent* power series in some region of interest, but if the function itself is well-behaved, it will have perfectly good polynomial approximations. These can be found by the methods of §5.8, but *not* by economization of series. There is slightly less to economization of series than meets the eye.

CITED REFERENCES AND FURTHER READING:

Acton, F.S. 1970, *Numerical Methods That Work*; 1990, corrected edition (Washington: Mathematical Association of America), Chapter 12.

Arfken, G. 1970, *Mathematical Methods for Physicists*, 2nd ed. (New York: Academic Press), p. 631. [1]

5.12 Padé Approximants

A *Padé approximant*, so called, is that rational function (of a specified order) whose power series expansion agrees with a given power series to the highest possible order. If the rational function is

$$R(x) \equiv \frac{\displaystyle\sum_{k=0}^{M} a_k x^k}{1 + \displaystyle\sum_{k=1}^{N} b_k x^k} \tag{5.12.1}$$